Acyclic chromatic index of triangle-free 1-planar graphs

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Abstract

An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G. It was conjectured that $\chi'_a(G) \leq \Delta(G) + 2$ for any simple graph G with maximum degree $\Delta(G)$. A graph is 1-*planar* if it can be drawn on the plane such that every edge is crossed by at most one other edge. In this paper, we prove that every triangle-free 1-planar graph G has an acyclic edge coloring with $\Delta(G) + 16$ colors.

Keywords: Acyclic edge coloring; Acyclic chromatic index; κ-deletion-minimal graph; 1-planar graph
MSC: 05C15

1 Introduction

All graphs considered in this paper are simple, undirected and finite. For a plane graph G, we use $F(G)$ to denote the face set of G. In a plane graph G, the degree of a face f, denoted $\deg(f)$, is the length of the boundary walk. A k-vertex, k^{-}-vertex and k^{+}-vertex is a vertex with degree k, at most k and at least k, respectively. Analogously, we can define a k-face, k^{-}-face and k^{+}-face.

An acyclic edge coloring of a graph G is a proper edge coloring such that every cycle is colored with at least three colors. In other words, the union of any two color classes induces a subgraph such that every component is a path. The acyclic chromatic index $\chi'_a(G)$ of a graph G is the least number of colors in an acyclic edge coloring of G.

By Vizing's theorem, the acyclic chromatic index of G has a trivial lower bound $\Delta(G)$. Fiamčík [5] stated the following conjecture in 1978, which is well known as Acyclic Edge Coloring Conjecture, and Alon et al. [2] restated it in 2001.

Conjecture 1. For any graph G, $\chi'_a(G) \leq \Delta(G) + 2$.

Alon et al. [1] proved that $\chi'_a(G) \leq 64\Delta(G)$ by using probabilistic method. Molloy and Reed [11] improved it to $\chi'_a(G) \leq 16\Delta(G)$. Ndreca et al. [12] improved the upper bound to $\lceil 9.62(\Delta(G) - 1)\rceil$. Recently, Esperet and Parreau [4] further improved it to $4\Delta(G) - 4$ by using the so-called entropy compression method. To my knowledge, the best known general bound is $\lceil 3.74(\Delta(G) - 1)\rceil$ due to Giotis et al. [7]. Alon et al. [2] proved that there is a constant c such that $\chi'_a(G) \leq \Delta(G) + 2$ for a graph G whenever the girth is at least $c\Delta \log \Delta$.

Regarding general planar graph G, Fiedorowicz et al. [6] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$, and Hou et al. [10] proved that $\chi'_a(G) \leq \max\{2\Delta(G) - 2, \Delta(G) + 22\}$. Recently, Basavaraju et al. [3] showed that $\chi'_a(G) \leq \Delta(G) + 12$, and Guan et al. [8] improved it to $\chi'_a(G) \leq \Delta(G) + 10$, and Wang et al. [18] further improved it to $\chi'_a(G) \leq \Delta(G) + 7$. The current best upper bound is $\Delta(G) + 6$ by Wang and Zhang [17].

A graph is 1-planar if it can be drawn on the plane such that every edge crosses at most one other edge. Obviously, the class of 1-planar graphs is a larger class than the one of planar graphs. The concept of 1-planar graph was introduced by Ringel [13] in 1965, while he simultaneously colored the vertices and faces of a plane graph such that any pair of adjacent/incident elements receive distinct colors.

The Acyclic Edge Coloring Conjecture was verified for the triangle-free planar graphs, see [14, 16]. Recently, Song and Miao [15] firstly studied the acyclic chromatic index of triangle-free 1-planar graphs, and gave the following result.

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Theorem 1.1 (Song and Miao [15]). If G is a triangle-free 1-planar graph, then $\chi'_a(G) \leq \Delta(G) + 22$.

Note that the upper bound $\Delta(G) + 22$ is far from the conjectured bound $\Delta(G) + 2$. In this paper, we improve the upper bound to $\Delta(G) + 16$, and we believe it can be further improved.

Theorem 1.2. If G is a triangle-free 1-planar graph, then $\chi'_a(G) \leq \Delta(G) + 16$.

2 Preliminary and structural results

Let $\mathbb{S}$ be a multiset and x be an element in $\mathbb{S}$. The *multiplicity* $\mathrm{mul}_{\mathbb{S}}(x)$ is the number of times x appears in $\mathbb{S}$. Let $\mathbb{S}$ and $\mathbb{T}$ be two multisets. The union of $\mathbb{S}$ and $\mathbb{T}$, denoted by $\mathbb{S} \uplus \mathbb{T}$, is a multiset with $\mathrm{mul}_{\mathbb{S}\uplus\mathbb{T}}(x) = \mathrm{mul}_{\mathbb{S}}(x) + \mathrm{mul}_{\mathbb{T}}(x)$. A graph G with maximum degree at most κ is *κ-deletion-minimal* if $\chi'_a(G) > \kappa$ and $\chi'_a(H) \leq \kappa$ for every proper subgraph H of G.

A *partial acyclic edge coloring* of G is an acyclic edge coloring of any subgraph of G. Let ϕ be a partial acyclic edge coloring of G. Let $\mathcal{U}_\phi(v)$ denote the set of colors used on the edges incident with v. Let $A_\phi(v) = \{1, 2, \dots, \kappa\} \setminus \mathcal{U}_\phi(v)$ and $A_\phi(uv) = \{1, 2, \dots, \kappa\} \setminus (\mathcal{U}_\phi(u) \cup \mathcal{U}_\phi(v))$. Let $\Upsilon_\phi(u, v) = \mathcal{U}_\phi(v) \setminus \{\phi(uv)\}$ and $W_\phi(u, v) = \{\, u_i \mid uu_i \in E(G) \text{ and } \phi(uu_i) \in \Upsilon(u, v)\,\}$. Notice that $W_\phi(u, v)$ may be not the same as $W_\phi(v, u)$. For simplicity, we will omit the subscripts if no confusion can arise.

An (α,β)-maximal bichromatic path with respect to ϕ is a maximal path whose edges are colored by α and β alternately. An (α,β, u, v)-critical path with respect to ϕ is an (α,β)-maximal bichromatic path which starts at u with α and ends at v with α. An (α,β, u, v)-alternating path with respect to ϕ is an (α,β)-bichromatic path starting at u with α and ending at v with β.

A color α is *available* for an edge e in G with respect to a partial edge coloring of G if none of the adjacent edges of e is colored with α. An available color α is *valid* for an edge e if assigning the color α to e does not result in any bichromatic cycle in G.

Fact 1 (Basavaraju et al. [3]). Given a partial acyclic edge coloring of G and two colors α,β, there exists at most one (α,β)-maximal bichromatic path containing a particular vertex v. □

Fact 2 (Basavaraju et al. [3]). Let G be a κ-deletion-minimal graph and uv be an edge of G. If ϕ is an acyclic edge coloring of $G - uv$, then no available color for uv is valid. Furthermore, if $\mathcal{U}(u) \cap \mathcal{U}(v) = \emptyset$, then $\deg(u) + \deg(v) = \kappa + 2$; if $|\mathcal{U}(u) \cap \mathcal{U}(v)| = s$, then $\deg(u) + \deg(v) + \sum_{w \in W(u,v)} \deg(w) \geq \kappa + 2s + 2$. □

We collect some structural lemmas on κ-deletion-minimal graphs, which are useful for our main result.

Lemma 1. If G is a κ-deletion-minimal graph, then G is 2-connected and $\delta(G) \geq 2$.

The following two lemmas deal with the local structures of the 2-vertices

Lemma 2 (Wang and Zhang [16]). Let G be a κ-deletion-minimal graph. If v is adjacent to a 2-vertex v_0 and $N_G(v_0) = \{w, v\}$, then v is adjacent to at least $\kappa - \deg(w) + 1$ vertices of degree at least $\kappa - \deg(v) + 2$. Moreover,

- (A) if $\kappa \geq \deg(v)+1$ and $wv \in E(G)$, then v is adjacent to at least $\kappa-\deg(w)+2$ vertices of degree at least $\kappa-\deg(v)+2$, and $\deg(v) \geq \kappa - \deg(w) + 3$;
- (B) if $\kappa \geq \Delta(G) + 2$ and v is adjacent to precisely $\kappa - \Delta(G) + 1$ vertices of degree at least $\kappa - \Delta(G) + 2$, then v is adjacent to at most $\deg(v) + \Delta(G) - \kappa - 3$ vertices of degree two and $\deg(v) \geq \kappa - \Delta(G) + 4$. □

Lemma 3 (Wang and Zhang [16]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$. If v_0 is a 2-vertex, then v_0 is adjacent to two $(\kappa - \Delta(G) + 4)^+$-vertices.

Wang and Zhang also gave the following local structure of the 3-vertices.

Lemma 4 (Wang and Zhang [16]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$ and v be a 3-vertex with $N_G(v) = \{w, v_1, v_2\}$. If $\deg(w) = \kappa - \Delta(G) + 2$, then G has the following properties:

- (a) there is exactly one common color at w and v for any acyclic edge coloring of $G - wv$. By symmetry, we may assume that the color on vv_1 is the common color;

(b) $\deg(v_1) = \Delta(G) \geq \deg(v_2) \geq \kappa - \Delta(G) + 3$;

(c) the edge wv is not contained in any triangle in G and w is adjacent to exactly one 3^--vertex;

(d) the vertex v_1 is adjacent to at least $\kappa - \deg(v_2) + 1$ vertices of degree at least $\kappa - \Delta(G) + 2$;

(e) the vertex v_2 is adjacent to at least $\kappa - \Delta(G)$ vertices of degree at least $\kappa - \deg(v_2) + 2$;

(f) the vertex v_2 is adjacent to at least $\kappa - \Delta(G) + 1$ vertices of degree at least four. □

Lemma 5 (Hou et al. [9])**.** If G is a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$, then every 3-vertex is adjacent to three $(\kappa - \Delta(G) + 2)^+$-vertices.

Lemma 6. If G is a κ-deletion-minimal graph with $\kappa \geq \Delta(G)+2$, then every vertex is adjacent to at least two 4^+-vertices.

Proof. Let w be a vertex with neighbors $w_0, w_1, \dots, w_{\tau-1}$. Suppose to the contrary that w is adjacent to at most one 4^+-vertex. By Lemma 2, no 2-vertex is adjacent to w. Let w_0 be a 3-vertex with neighbors w, v_1, v_2. Since G is κ-deletion-minimal, the graph $G - ww_0$ has an acyclic edge coloring ϕ with $\phi(ww_i) = i$ for $1 \leq i \leq \tau - 1$. Note that $\deg(w) + \deg(w_0) = \deg(w) + 3 \neq \kappa + 2$, Fact 2 guarantees $|\mathcal{U}(w) \cap \mathcal{U}(w_0)| \geq 1$. Without loss of generality, we may assume that w_0v_1 is colored with 1.

Case 1. $|\mathcal{U}(w) \cap \mathcal{U}(w_0)| = 1$**.** Note that G cannot be acyclically edge colored with κ colors, thus there exists a $(1, \alpha, w, w_0)$-critical path for $\alpha \in A(ww_0)$, and then $A(ww_0) \subseteq \Upsilon(w, w_1) \cap \Upsilon(w_0, v_1)$. In this case, we consider the following two situations according to the degree of w_1.

Subcase 1.1. w_1 **is a 3-vertex.** Recall that there exists a $(1, \alpha, w, w_0)$-critical path for $\alpha \in A(ww_0)$, thus $\tau = \kappa - 2 = \Delta$ and $\Upsilon(w, w_1) \subseteq \{\Delta, \Delta + 1, \Delta + 2\}$. If there exists another vertex w_s with $\Upsilon(w, w_s) \subseteq \{\Delta, \Delta + 1, \Delta + 2\}$, then we can exchange the colors on ww_1 and ww_s, and additionally color ww_0 with an element in $A(ww_0)$. Hence, we have $\Upsilon(w, w_s) \cap \{1, 2, \dots, \Delta - 1\} \neq \emptyset$ for $s \geq 2$. Let $w_2, w_3, \dots, w_{\tau-2}$ be 3-vertices. For $i \geq 2$, uncolor ww_i and color ww_0 with i, we obtain an acyclic edge coloring ϕ_i of $G - ww_i$.

There exists a $(\lambda_\alpha, \alpha, w, w_2)$-critical path for each α in $A(ww_2)$, for otherwise we can color ww_0 with 2 and recolor ww_2 with α. It follows that there exists $x, y \in A(ww_2)$ with $\lambda_x = \lambda_y = \lambda$, and then $\{x, y\} \subseteq \Upsilon(w, w_\lambda)$. By Fact 1, we have that $\lambda \neq 1$, and then the vertex w_λ is a 4^+-vertex and $\lambda = \tau - 1$.

By similar arguments, we conclude that there exists a $(\tau - 1, \alpha_1, w, w_3)$-critical path and a $(\tau - 1, \alpha_2, w, w_3)$-critical path for some α_1, α_2 in $A(ww_3)$. Note that $\{\alpha_1, \alpha_2\} \cup \{x, y\} \subseteq \{\Delta, \Delta + 1, \Delta + 2\}$, thus $\{\alpha_1, \alpha_2\} \cap \{x, y\} \neq \emptyset$, but this contradicts Fact 1.

Subcase 1.2. w_1 **is a** 4^+**-vertex.** Note that $|A(v_1) \cap \{2, 3, \dots, \tau - 1\}| \geq 1$, otherwise $\deg(v_1) \geq \kappa - 1 \geq \Delta + 1$. By symmetry, we may assume that 2 is a missing color at v_1. Uncolor ww_2 and color ww_0 with 2, the resulting coloring is an acyclic edge coloring φ of $G - ww_2$. By Fact 2, we have that $\Upsilon(w, w_2) \cap \{1, 2, \dots, \tau - 1\} \neq \emptyset$.

- $\Upsilon(w, w_2) = \{\rho, \rho'\}$ and $\rho' \geq \tau$. There exists a (ρ, α, w, w_2)-critical path for $\alpha \in A(ww_0) \cap A(ww_2)$, thus $\rho \neq 1$ and w_ρ is a 3-vertex. Now, we can reduce it to Subcase 1.1 with φ playing the role of ϕ.

- $\Upsilon(w, w_2) = \{\rho, \rho'\} \subseteq \{1, 2, \dots, \tau - 1\}$. Note that none of $\tau, \tau + 1, \dots, \kappa$ is valid for ww_2 under φ, thus there exists a (ρ, α_1, w, w_2)-critical path and a (ρ, α_2, w, w_2)-critical path for some α_1, α_2 from $\{\tau, \tau + 1, \dots, \kappa\}$. It is obvious that $\{\alpha_1, \alpha_2\} \cap A(ww_0) \neq \emptyset$, thus $\rho \neq 1$ and $\Upsilon(w, w_\rho) = \{\alpha_1, \alpha_2\}$. So we may assume that $\rho = 3$.

  If $3 \notin \mathcal{U}(v_1)$, then we can color ww_0 with 3 and recolor ww_3 with an element in $\{\tau, \tau + 1, \dots, \kappa\} \setminus \{\alpha_1, \alpha_2\}$. It follows that $3 \in \mathcal{U}(v_1)$.

  Suppose that $4 \notin \mathcal{U}(v_1)$ and $\Upsilon(w, w_4) = \{p, q\}$. For each $\alpha \in A(ww_4)$, there exists a (p, α, w, w_4)-critical path or a (q, α, w, w_4)-critical path, for otherwise we can color ww_0 with 4 and recolor ww_4 with α.

  – Suppose that $q \geq \tau$. Thus there exists a (p, α, w, w_4)-critical path for each $\alpha \in A(ww_4)$, and then $p \neq 3$ and $A(ww_4) \subseteq \Upsilon(w, w_p)$. Note that $A(ww_0) \cap A(ww_4) \neq \emptyset$, so we have that $p \neq 1$. In fact w_p is a 3-vertex and $\Upsilon(w, w_p) = A(ww_4)$. We can exchange the colors on ww_3 and ww_p, color ww_0 with 4 and recolor ww_4 with an element in $A(ww_4)$.

– Suppose that $\{p, q\} \subseteq \{1, 2, \ldots, \tau - 1\}$. Note that $A(ww_4) = \{\tau, \tau + 1, \ldots, \kappa\}$ and $|A(ww_4)| \geq 3$, so we may assume that there exists a (p, ξ_1, w, w_4)-critical path and a (p, ξ_2, w, w_4)-critical path for some $\xi_1, \xi_2 \in A(ww_4)$. It concludes that $\{\xi_1, \xi_2\} \subseteq \Upsilon(w, w_p)$. Clearly, $\{\xi_1, \xi_2\} \cap A(ww_0) \neq \emptyset$, and then $p \in \{1, 3\}$ due to Fact 1. So w_p is a 3-vertex with $\Upsilon(w, w_p) = \{\xi_1, \xi_2\}$. We can exchange the colors on ww_3 and ww_p, color ww_0 with 2 and uncolor ww_2, and then we obtain a new acyclic edge coloring of $G - ww_2$. Under this new coloring, there exists a (ρ', α, w, w_2)-critical path for each $\alpha \in A(ww_2)$, and then $A(ww_2) \subseteq \Upsilon(w, w_{\rho'})$. If $\rho' = 1$, then there exists a $(1, \alpha, w, w_2)$-critical path and a $(1, \alpha, w, w_0)$-critical path for each $\alpha \in A(ww_0)$, which contradicts Fact 1. If $\rho' \neq 1$, then $\deg(w_{\rho'}) \geq 1 + |A(ww_2)| \geq 4$, a contradiction.

By similar arguments, $\{4, 5, \ldots, \tau - 1\} \subseteq \mathcal{U}(v_1)$, and then $\{1, 3, 4, \ldots, \tau - 1\} \cup A(ww_0) \subseteq \mathcal{U}(v_1)$. It follows that $A(v_1) = \{2, \phi(w_0v_2)\}$. We recolor w_0v_1 with 2, and then we reduce it to Subcase 1.1.

Case 2. $|\mathcal{U}(w) \cap \mathcal{U}(w_0)| = 2$. By symmetry, we may assume that w_0v_2 is colored with 2. There exists a $(1, \alpha, w_0, w)$-critical path or a $(2, \alpha, w_0, w)$-critical path for $\alpha \in \{\tau, \tau + 1, \ldots, \kappa\}$, thus $\{\tau, \tau + 1, \ldots, \kappa\} \subseteq \Upsilon(w, w_1) \cup \Upsilon(w, w_2)$.

Subcase 2.1. Either $\Upsilon(w_0, v_1) \not\supseteq \{\tau, \ldots, \kappa\}$ or $\Upsilon(w_0, v_2) \not\supseteq \{\tau, \ldots, \kappa\}$. By symmetry, we may assume that $\tau \notin \Upsilon(w_0, v_2)$. Note that τ is not valid for ww_0, thus there exists a $(1, \tau, w_0, w)$-critical path, and then there exists no $(1, \tau, w_0, v_2)$-critical path. Recoloring w_0v_2 with τ results in a new acyclic edge coloring σ of $G - ww_0$ with $|\mathcal{U}_\sigma(w) \cap \mathcal{U}_\sigma(w_0)| = 1$ and it takes us back to Case 1.

Subcase 2.2. $\Upsilon(w_0, v_1) \supseteq \{\tau, \ldots, \kappa\}$ and $\Upsilon(w_0, v_2) \supseteq \{\tau, \ldots, \kappa\}$.

$(*_1)$ Note that at most one of w_1 and w_2 is a 4^+-vertex, so we may assume that w_2 is a 3-vertex. If $\Upsilon(w, w_2) \subseteq \{\tau+1, \tau+2, \ldots, \kappa\}$, then we can recolor ww_2 with an element in $\{\tau, \tau+1, \ldots, \kappa\} \setminus \Upsilon(w, w_2)$, and then reduce it to Case 1. So we may assume that $\Upsilon(w, w_2) = \{\rho, \rho'\}$ with $\rho < \tau$. There exists a $(1, \alpha, w, w_0)$-critical path for $\alpha \in \{\tau, \tau+1, \ldots, \kappa\} \setminus \{\rho'\}$, thus $\{\tau, \tau+1, \ldots, \kappa\} \setminus \{\rho'\} \subseteq \Upsilon(w, w_1)$. If w_1 is a 3-vertex, then $\Upsilon(w, w_1) \cup \{\rho'\} = \{\tau, \tau+1, \ldots, \kappa\} = \{\Delta, \Delta+1, \Delta+2\}$, and then we can recolor ww_1 with an element in $\{\tau, \tau+1, \ldots, \kappa\} \setminus \Upsilon(w, w_1)$ and reduce it to Case 1. Hence, w_1 is a 4^+-vertex.

$(*_2)$ Since $A(v_1) \cap \{3, 4, \ldots, \tau - 1\} \neq \emptyset$, we may assume that 3 is a missing color at v_1. Recoloring w_0v_1 with 3 must create a $(3, 2)$-bichromatic cycle containing w_0v_1, for otherwise the resulting coloring is a new acyclic edge coloring of $G - ww_0$, and then one of w_2 and w_3 must be a 4^+-vertex by a similar argument in the last paragraph. Let σ be obtained by uncoloring ww_3 and coloring ww_0 with 3. It is obvious that σ is an acyclic edge of $G - ww_3$. We can conclude that $\Upsilon(w, w_3) \subseteq \{1, 2, \ldots, \tau - 1\}$, otherwise we reduce it to Case 1.

$(*_3)$ Let $\Upsilon(w, w_3) = \{p, q\}$. By a similar argument as in the paragraph marked with $(*_1)$, one of w_p and w_q is a 4^+-vertex. So we may assume that $p = 1$. For $\alpha \in \{\tau, \tau + 1, \ldots, \kappa\} \setminus \{\rho'\}$, there exists no $(1, \alpha, w, w_3)$-critical path, so there exists a (q, α, w, w_3)-critical path and $\{\tau, \tau + 1, \ldots, \kappa\} \setminus \{\rho'\} \subseteq \Upsilon(w, w_q)$. Since w_q is a 3-vertex, thus $\Upsilon(w, w_q) \cup \{\rho'\} = \{\tau, \tau+1, \ldots, \kappa\} = \{\Delta, \Delta+1, \Delta+2\}$. Hence, there exists a $(1, \rho', w, w_3)$-critical path, for otherwise we can color ww_0 with 3 and recolor ww_3 with ρ'.

Recall that recoloring w_0v_1 with 3 creates a $(3, 2)$-bichromatic cycle containing w_0v_1, this implies that $2 \in \mathcal{U}(v_1)$ and $|A(v_1) \cap \{3, 4, \ldots, \tau - 1\}| \geq 2$. So we may assume that 4 is also a missing color at v_1. By a similar argument as in the paragraphs marked with $(*_2)$ and $(*_3)$, there exists a $(1, \rho', w, w_4)$-critical path, but this contradicts Fact 1. □

In [17], Wang and Zhang presented the following structural lemma of the 4-vertices.

Lemma 7 (Wang and Zhang [17]). Let G be a κ-deletion-minimal graph with $\kappa \geq \Delta(G) + 2$, and let w_0 be a 4-vertex with $N_G(w_0) = \{w, v_1, v_2, v_3\}$.

(A) If $\deg_G(w) \leq \kappa - \Delta(G)$, then

$$\sum_{x \in N_G(w_0)} \deg_G(x) \geq 2\kappa - \deg_G(w_0) + 8 = 2\kappa + 4. \tag{1}$$

(B) If $\deg_G(w) \leq \kappa - \Delta(G) + 1$ and ww_0 is contained in two triangles, then

$$\sum_{x \in N_G(w_0)} \deg_G(x) \geq 2\kappa - \deg_G(w_0) + 9 = 2\kappa + 5. \tag{2}$$

Furthermore, if the equality holds in (2), then all the other neighbors of w are 6^+-vertices. □

3 Proof of Theorem 1.2

Now, we are ready to prove the main result in this paper.

Proof of Theorem 1.2. Suppose that G is a counterexample to the theorem in the sense that $|V| + |E|$ is minimum. It is easy to see that G is a κ-deletion-minimal graph, where $\kappa := \Delta(G) + 16$. By Lemma 1, the graph G is 2-connected and $\delta(G) \geq 2$.

Since G is κ-deletion-minimal, it has the following local structures.

(C1) Every 2-vertex is adjacent to two 20^+-vertices (Lemma 3).

(C2) Every 3-vertex is adjacent to three 18^+-vertices (Lemma 5).

(C3) Every 18-vertex is adjacent to at most one 3-vertex (Lemma 4).

(C4) Every vertex is adjacent to at least two 4^+-vertices (Lemma 6).

(C5) Every 4-vertex is adjacent to four 10^+-vertices, or a 9^--vertex and three 22^+-vertices (Lemma 7 (A)).

Suppose that G contains a 5-vertex v adjacent to three 7^--vertices. Let $N_G(v) = \{u, v_1, v_2, v_3, v_4\}$ with $\deg(u) \leq 7, \deg(v_1) \leq 7$ and $\deg(v_2) \leq 7$. By the minimality of G, the graph $G - uv$ has an acyclic edge coloring ϕ with $\Delta(G) + 16$ colors. Moreover, when we choose the acyclic edge coloring ϕ, we assume that the number of common colors on the edges incident with u and v is minimum, that is, $|\mathcal{U}(u) \cap \mathcal{U}(v)| = m$ is minimum among all the acyclic edge colorings of $G - uv$. We can easily obtain that $m \geq 1$ from Fact 2. Let $N_G(u) = \{v, u_1, \dots, u_t\}, t \leq 6$.

The first case: $m = 1$. Assume that uu_1 and vv_1 have the same color 1. Note that all the available colors for uv are invalid, hence there exists a $(1, \alpha, u, v)$-critical path for each α in $A(uv)$, and thus $A(uv) \subseteq \mathcal{U}(u_1)$. But $|A(uv)| \geq \kappa - (6 + 4 - 1) > \Delta$, thus $|\mathcal{U}(u_1)| \geq |A(uv)| + 1 > \Delta$, a contradiction.

The second case: $m \geq 2$. Assume that uu_i and vv_i have the same color i for each $i \in \{1, 2, \dots, m\}$. For each $\alpha \in A(uv)$, there exists an (i_α, α, u, v)-critical path for some $i_\alpha \in \{1, 2, \dots, m\}$. Note that $|A(uv)| \geq \kappa - 10 + m \geq \kappa - 8 \geq \Delta + 8$,

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) - 4 < 2|A(uv)|,$$

thus there exists an available color α^* such that it appears exactly once in $\mathbb{S}$, where $\mathbb{S}$ is defined as $\mathbb{S} := \Upsilon(v, v_1) \uplus \Upsilon(v, v_2) \uplus \Upsilon(v, v_3) \uplus \Upsilon(v, v_4)$. Without loss of generality, we may assume that it appears in $\mathcal{U}(v_1)$, and then there exists a $(1, \alpha^*, u, v)$-critical path. Now, we revise ϕ by recoloring vv_2 with α^*, which yields a new acyclic edge coloring of $G - uv$, but it contradicts the minimality of m. Therefore, the graph G does not contain a 5-vertex adjacent to three 7^--vertices. That is,

(C6) every 5-vertex is adjacent to at least three 8^+-vertices.

Discharging Part. In the following, we may assume that G has been embedded on the plane such that every edge is crossed by at most one other edge. Moreover, the number of crossings is as small as possible. We treat each of the crossings as a vertex and obtain an *associated plane graph* $G^\dagger$.

Since the number of crossings is as small as possible in the embedding, we have the following claim.

Claim 1. Every 2-vertex is incident with two 4^+-faces in $G^\dagger$.

Since G is triangle-free and every 2-vertex is incident with two 4^+-faces in $G^\dagger$, we have the following statement. A similar statement has been proven in [15].

Claim 2. Every ℓ-vertex is incident with at most $\left\lfloor \frac{2(\ell-\lambda)}{3} \right\rfloor$ 3-faces in $G^\dagger$, where λ is the number of adjacent 2-vertices.

We use the discharging method to derive a contradiction. Here, we need the following rewritten Euler's formula for the associated plane graph $G^{\dagger}$:

$$\sum_{v \in V(G^{\dagger})} (\deg(v) - 4) + \sum_{f \in F(G^{\dagger})} (\deg(f) - 4) = -8. \tag{3}$$

At first, we assign the initial charge of every vertex v to be $\deg(v) - 4$ and the initial charge of every face f to be $\deg(f) - 4$. Next, we design appropriate discharging rules and redistribute charges among vertices and faces, such that the final charge of every vertex and every face is nonnegative, which leads to a contradiction. Note that all the adjacencies between vertices in the discharging rules are refer to the adjacencies between vertices in G, not in $G^{\dagger}$.

Discharging rules:

(R1) every 2-vertex receives 1 from each adjacent vertex;

(R2) every 3-vertex receives $\frac{3}{2}$ from the adjacent 18-vertex;

(R3) every 3-vertex receives $\frac{1}{2}$ from each adjacent 19^+-vertex;

(R4) every 3-vertex receives $\frac{1}{2}$ from each incident 5^+-face;

(R5) every 3-face receives $\frac{1}{2}$ from each incident non-crossing vertex;

(R6) every non-crossing 4-vertex receives $\frac{1}{4}$ from each adjacent vertex if it is adjacent to four 10^+-vertices;

(R7) every non-crossing 4-vertex receives $\frac{1}{3}$ from each adjacent 22^+-vertex if it is adjacent to a 9^--vertex and three 22^+-vertices;

(R8) every 5-vertex receives $\frac{1}{6}$ from each adjacent 8^+-vertex.

If w is an arbitrary 2-vertex, then its final charge is $2 - 4 + 2 \times 1 = 0$. Let w be an arbitrary 3-vertex. Note that w is incident with at most two 3-faces. If w is incident with at most one 3-face, then its final charge is at least $3 - 4 + \frac{3}{2} - \frac{1}{2} = 0$. On the other hand, if w is incident with exactly two 3-faces, then it is incident with a 5^+-face and it receives $\frac{1}{2}$ from the 5^+-face, and then its final charge is at least $3 - 4 + \frac{3}{2} + \frac{1}{2} - 2 \times \frac{1}{2} = 0$. Hence, the final charge of an arbitrary 3-vertex is nonnegative.

It is obvious that the final charge of a crossing 4-vertex is zero. Let w be an arbitrary non-crossing 4-vertex. If w is adjacent to four 10^+-vertices, then its final charge is at least $4 - 4 + 4 \times \frac{1}{4} - 2 \times \frac{1}{2} = 0$. If w is adjacent to a 9^--vertex and three 22^+-vertices, then its final charge is at least $4 - 4 + 3 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0$.

If w is a 5-vertex, then it is adjacent to at least three 8^+-vertices, then its final charge is at least $5-4+3\times\frac{1}{6}-3\times\frac{1}{2} = 0$.

If w is an arbitrary ℓ-vertex with $\ell = 6, 7$, then its final charge is at least $\ell - 4 - \frac{2\ell}{3} \times \frac{1}{2} \geq 0$.

If w is an arbitrary ℓ-vertex with $\ell = 8, 9$, then its final charge is at least $\ell - 4 - \frac{2\ell}{3} \times \frac{1}{2} - \frac{1}{6}\ell \geq 0$.

Let w be an arbitrary 10^+-vertex in the following. Suppose that w is adjacent to at least one 2-vertex. Let λ be the number of adjacent 2-vertices. By Lemma 2, it is adjacent to at least seventeen 18^+-vertices, thus its final charge is at least $\ell - 4 - \frac{2(\ell-\lambda)}{3} \times \frac{1}{2} - \lambda \times 1 - (\ell - \lambda - 17) \times \frac{1}{2} = \frac{\ell - \lambda}{6} + \frac{9}{2} > 0$. So we may assume that w is not adjacent to any 2-vertex.

- If w is an ℓ-vertex with $\ell \geq 22$, then its final charge is at least $\ell - 4 - \left\lfloor \frac{2\ell}{3} \right\rfloor \times \frac{1}{2} - \ell \times \frac{1}{2} \geq 0$.
- If w is an ℓ-vertex with $\ell = 19, 20, 21$, then its final charge is least $19 - 4 - \left\lfloor \frac{2\times 19}{3} \right\rfloor \times \frac{1}{2} - 17 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$, $20 - 4 - \left\lfloor \frac{2\times 20}{3} \right\rfloor \times \frac{1}{2} - 18 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$, or $21 - 4 - \left\lfloor \frac{2\times 21}{3} \right\rfloor \times \frac{1}{2} - 19 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$.
- If w is an 18-vertex, then it is adjacent to at most one 3-vertex, and then its final charge is at least $18 - 4 - \frac{2\times 18}{3} \times \frac{1}{2} - \frac{3}{2} - 17 \times \frac{1}{4} > 0$.
- If w is an ℓ-vertex with $10 \leq \ell \leq 17$, then it is only adjacent to 4^+-vertices, and then its final charge is at least $\ell - 4 - \frac{2\ell}{3} \times \frac{1}{2} - \ell \times \frac{1}{4} > 0$.

It is obvious that every 3-face has the final charge $3 - 4 + 2 \times \frac{1}{2} = 0$. Every 4-face has the final charge as its initial charge, zero. By (C2), there is no consecutive 3-vertices lying on a face boundary, thus every 5^+-face f has final charge at least $\deg(f) - 4 - \left\lfloor \frac{\deg(f)}{2} \right\rfloor \times \frac{1}{2} \geq 0$.

Now, the final charge of every vertex and every face is nonnegative, which derives the desired contraction. □

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